An Investigation into the Assertion Sign in Principia Mathematica

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Abstract.

I am attempting to ascertain the meaning of the assertion sign "|-" in Principia Mathematica. Some discussion of it occurs on page 92 of the edition to *56. Gregory Landini and I had a disagreement about this in an email discussion. He maintained that it means "it is a theorem that." PM to *56 on p. 92 says that it may be read "it is true that." It is followed by the parenthetical qualification "although philosophically this is not exactly what it means." So, I am trying to determine more exactly what it means.

I develop an idea that appears in Russell's 1906 work notes "On Substitution." Russell's motivation, is to provide an account of false belief and to develop what he calls a "multiple-relation theory." Instead of taking "*A* believes φ(*a*, *b*)" to indicate a dyadic relation of 'belief' between person A and an entity 'φ(*a*, *b*),' we find that Russell considered a relation of ideas Φ_i that holds between i`*a* which is an idea A has of *a* and i`*b* which is an idea A has of *b*. Similarly, "A asserts that φ*(a, b)*" is to be taken as involving a relation Φ_n that holds between n`*a* which is a name A has of *a* and n`*b* which is a name A has of *b*. Belief involves ideas before the mind, while assertions involve names (or words).

My foremost reason for disagreeing with Landini is that it would not permit PM to justify the following reasoning:

|- Caesar died.

|- Caesar died ⊃ Caesar is dead.

therefore

|- Caesar is dead.

I believe this deduction is justified by *1.1 Anything implied by a true elementary proposition is true. This is also supported by the discussion in PM, pp. 8-9. It is also supported by the discussion of Lewis Carroll's "What the Tortoise said to Achilles", in section 38 of Russell's Principles of Mathematics, which is referred to on pages 92 and 94 of PM.

None of these are theorems, though I think we want to assert they are true, and even if they are not true, that would not make the argument invalid, only unsound. Of course, Whitehead and Russell wanted all their theorems to be true. As it is a work of logic, they only asserted theorems. That does not mean they did not think nothing other than theorems could be asserted. Landini often seems to anachronistically apply concepts developed since PM was written to PM. One example being the concept of well formed formulas (wff).

Notation.

I use notation from Principia Mathematica to *56 [hereafter PM] as much as possible. Backward apostrophe is PM's notation for *function application* (PM *30.01). sin`x would, in usual algebra, be written sin(x). In "On Substitution", i is a function from objects to ideas of the objects. n is a function from objects to their names. The *relative product* (PM *34.01) of two relations R and S is written R|S and defined by

$$x(R|S)y \ =\text{df. } (\exists z)(\, R(x, z) \bullet S(z, y)).$$

Cnv is the relation between a relation and its converse (PM *31.01), so Cnv`R is the converse of the relation R.

I introduce a special case of definite descriptions - *14.01

$((\iota\,\varphi)(S(f, \varphi))\,\{(\iota\,x)S(u, x), (\iota\,y)S(v, y)\}\ \ =df$ *14.01

$(\exists g)((z)(S(f, z) \equiv z = g)) \bullet (\exists c)((a)(S(u, a) \equiv a = c)) \bullet (\exists d)((b)(S(v, b) \equiv b = d)) \bullet g\{c,d\}.$

There are three entities described, (ιφ), (ιa) and (ιb). They may be taken to be the g, c, and d such that g is the relational object (predicate) such that f is uniquely related to it by the S relation. Please notice that this (ι φ)(S(f, φ) is a definite description in predicate position – something not done in PM as far as I know, but it is defined in terms PM uses with predicates – quantification and identity (of g). c is the object such that u is uniquely related to it by the S relation, and d is the object such that v is uniquely related to it by the S relation. And g(c, d). Explained more straightforwardly (I hope), f is the idea of g, i.e. (S(f, g)) , and g is the only such entity. I am assuming S to be a many-one relation. That amounts to assuming our ideas to be unambiguous. Also, u is the idea of c, i.e. S(u, c); and c is the only object in that relation. Similarly, v is the idea of d and d is the only object in that relation. And the g relation holds between c and d.

In summary:

$(\exists\, g)\ (z)\ (S(f, z) \equiv (z = g))$

Assures that there is one and only one such z and it is g.

$(\exists\, c)\ (a)\ (S(u, a) \equiv (a = c))$

Assures that there is one and only one such a and it is c.

$(\exists\, d)\ (b)\ (S(v, b) \equiv (b = d))$

Assures that there is one and only one such b and it is d.
Note that this is PM *14.01 applied 3 times. f, u and v are values – being ideas that are determined previously in the overall statement. In each of the S(idea, object), should be regarded as a function of the object. S(idea, x) is φx in *14.01. The idea is regarded as fixed previously.

Then having required the existence and uniqueness of g, c, and d, we require in addition that g(c, d).

If the existence or uniqueness of any of the g, c, or d fail the statement is false. It is also false, in addition, if g(c, d) is not true.

It is perhaps easier to understand with

$((\iota\, \varphi)(S(f, \varphi))\ \{(\iota\, x)S(u, x), (\iota\, y)S(v, y)\}\ \ =df$

$(\exists\, g)((\iota\, \varphi)(S(f, \varphi)) = g) \bullet (\exists\, c)((\iota\, x)S(u, x) = c) \bullet (\exists\, d)((\iota\, y)S(v, y) = d)$
$\bullet\, g\{c,d\}$

Introduction.

I am attracted to an approach that Russell experimented with in his work notes "On Substitution" (CPBR, vol 5, paper 5a, p. 185ff) [hereafter OS] and I had independently developed something similar, and was pleased to notice Russell had anticipated these ideas in his work notes. On this theory, for a person A at time t, there is a relation *S* involved in belief that is between ideas and the entities they are ideas of. In Assertion there is a relation *R* involved between names (words) and the corresponding ideas. These relations are primitive and unanalyzable, but in words we can say the follows:

$S^A(x, y)$ =df. *x* is an idea A has of *y*
$R^A(x, y)$ =df. *x* is a name A has of the idea *y*

If we add a temporal component t, we can write:

$S^{A,t}(x, y)$ =df. at time t, *x* is an idea A has of *y*
$R^{A,t}(x, y)$ =df. at time t, *x* is a name A has of the idea *y*

It is convenient to use terms such as "ia" for an idea of *a*, "na" for a name of *a*. With this convention we naturally think the following must hold:

$R^A(na, ia)$ $S^A(ia, a)$
$R^A(nb, ib)$ $S^A(ib, b)$
$R^A(nP, iP)$ $S^A(iP, P)$

On my view, the relations *R* and *S* are objective and cannot be assured to hold merely by the psychological will of a person. For a person A to have a belief, the person must form the relevant ideas which are involved in the belief, but in some cases, the *R* or *S* relations (or both) may fail to hold. If the *S* relation fails to hold then what is formed is not a what I call a "full belief" since what is formed is unduly private and psychological. For this I introduce

$B_{private}(A, x, y, z)$

It is assumed that in such a case, *x*, *y*, *z* are all ideas formed by A. It is left open whether any of the ideas in question are ideas of entities which exist. But if, for example S(*x*, *a*), then *x* is an idea of *a*, and it is assured thereby that *a* exists. One can have an idea that doesn't stand in the S relation to anything. It is in this sense alone that one can be said to "have an idea of something that doesn't exist." Similarly, if the *R* relation fails to hold, then an assertion of the belief is not possible. (it should be noted that Russell's sketch in "On Substitution," does not have this feature.)

My Relations defined in terms of Russell's Relations.

Russell used two relations i and n in "On Substitution."
In defining R and S in Russell's terms:

$\mathrm{R}^{\mathrm{A}}(\mathrm{na}, \mathrm{ia}) \equiv (\mathrm{na} = \mathrm{n`a} \bullet \mathrm{ia} = \mathrm{i`a})$
$\mathrm{S}^{\mathrm{A}}(\mathrm{ia}, \mathrm{a}) \equiv (\mathrm{ia} = \mathrm{i`a})$

The difference between Russell and me is he takes objects as definitive. Then there can be no definition of ideas or names for which there is no object.

Russell's Relations defined in terms of my Relations.

Defining Russell's terms in of R and S:

$n`a = na \equiv (R^A(na, ia) \bullet S^A(ia, a))$
$i`a = ia \equiv S^A(ia, a)$
This amounts properly to:
n =df. Cnv`S | Cnv`R
i =df. Cnv`S

Here Cnv is the Converse of a relation.

As I take ideas primitive, there can be beliefs about ideas for which no object exists. As Russell takes objects as primitive, he cannot describe ideas with no corresponding object.

$\mathrm{B}_{private}$ is a primitive relation of a person to his ideas

In addition to R and S, I also take there to be a primitive psychological relation that may hold between a person A's ideas – even if no corresponding objects exist. I call it $\mathrm{B}_{private}$.

With the *R* and the *S* relations now in place, one can observe that the relative product of those relations, namely R|S is a relation between a name (word) and an entity. That is because we have:

$x(R|S)y$ =df. $(\exists z)(R(x, z) \bullet S(z, y))$.

There may be objects for which a person A has no ideas, but in such a case A could not form singular beliefs about those specific objects.

Also, there may be objects for which a person A has no words and thus A cannot make singular assertions about those objects. Now I have the following:

Full Belief defined.

$\mathrm{B}_{private}$ and S are primitives.

A believes that *φ(a, b) =df.*

(∃x, y, z) $B_{private}$*(A, φ, a, b)* • S^A *(a, x))* • S^A *(b, y)))* • S^A *(φ, z)*

In the above, ϕ, a and b are A's ideas before his mind as so guaranteed to exist since A has formed the belief. x, y, and z are variables – there may be no values that satisfy the relations.

True Belief.

In addition, R is a primitive.

A believes truly that *φ(a, b) =df.*

$B_{private}$*(A, φ, a, b)* • *((ιz)(* S^A*(φ, z))) {(ιx) (*S^A*(a, x)), (ιy)(* S^A*(b, y)}*

In the above, ϕ, a and b are person A's ideas before his mind as so guaranteed to exist since he has formed the belief. The truth of the belief is given by a predicate and its arguments - all given by definite descriptions. If the descriptions fail then there is no true belief. The definition of this form of definite descriptions is given in the section on notation above. Note: in *((ι z)(* S^A*(φ, z))) {(ι x) (*S^A*(a, x)), (ι y)(* S^A*(b, y)}* the described predicate is applied to two described objects.

False Belief.

Person A believes falsely if person A believes but does not believe truly. Person A believing falsely is not the same as it being false that person A believes.

A believes falsely that *φ(a, b) =df.*
$B_{private}$*(A, φ, a, b)* • S^A *(a, x))* • S^A *(b, y)))* • S^A *(φ, z)*
• ~ *((ι z)(* S^A*(φ, z))) {(ι x) (*S^A*(a, x)), (ι y)(* S^A*(b, y)}*

Note: *((ι z)(* S^A*(φ, z))) {(ι x) (*S^A*(a, x)), (ι y)(* S^A*(b, y)}*
Uses the described predicate applied to two described objects. The descriptions must succeed for there to be belief, but if described relation fails, then the belief is false.

It is false that A believes *φ(a, b) =df.*
~ *((∃x, y, z)* $B_{private}$*(A, φ, a, b)* • S^A *(a, x))* • S^A *(b, y)))* • S^A *(φ, z))*

It can be true if either the private belief does not exist or any of the objects fail to correspond to A's ideas.
Yet another sort of negative is A believes ~ *φ(a, b)*

Assertion defined.

We are now ready to define assertion, which is distinct from belief. I take "saying" as a primitive relation between names (i.e. words).

A asserts that *Φ (a, b) =df.*
Says(A, Φ ,a, b) • *(∃w, u, v)* R^A *(Φ, w)* • R^A *(a, u)* • R^A *(b,v)*

Φ , a, and b are words in the assertion – so they exist if person A made the assertion (said it). There will be w, u and v if the words have meaning to person A. If there are no such [ideas] w, u, or v then the existential quantifier fails and there is no assertion. Also, perhaps, the ideas must be limited to ones which make sense when so combined. That is beyond the scope of this paper and will not be considered. There may be no objects [corresponding to the ideas] with Cnv`S relations to w, u, or v. In that case, there still can be such an assertion. That is A may use words which correspond to ideas – but ideas to which no objects correspond. Yet, an assertion is made. Russell had a problem in this case as, for him, the words are described in terms of the objects. He has no relation between words and ideas (R), if there is no object. It would be

$R^A(x, y) \equiv (\exists a)(x = n`a \bullet y = i`a).$

But that definition of the relation requires the object a to exist. The R relation I am using does not require there to be such an object. We may have both words and ideas to which no object corresponds. We are considering that there may be no such objects. This is possibly important later in the case of the liar.

True Assertion.

A asserts truly that $\Phi\ (a, b)$ =df.

$Says(A, \Phi, a, b) \bullet (\exists w, u, v)\ R^A(\Phi, w)) \bullet R^A(a, u)) \bullet R^A(b,v)$
$\bullet ((\iota z)(S^A(w, z)))\ \{(\iota x)\ S^A(u, x), (\iota y)\ S^A(v, y)\}$

This is different. To be true, there must be objects with the Cnv`S relations to the ideas φ, x, or y. We also need $((\iota z)(S^A(w, z)))\ \{(\iota x)\ S^A(u, x), (\iota y)\ S^A(v, y)\}$. This was defined in the section on notation, and also implies that those objects exist. Note you can assert things that you do not believe. Honest assertion would add $B_{private}$.

The objects are specified by definite descriptions, so the assertion makes sense (although false) if the objects do not exist. Person A does not assert truly if any of the objects z, x, or y [I am actually speaking of the corresponding descriptions] do not exist, but person A does assert, by the previous definition. There cannot be a true assertion without *z, x* or *y*. If any the ideas *w, u*, or *v* do not exist then there is no assertion – some of A's words, Φ *,a, or b* have no meaning for A.

False Assertion.

Person A asserts falsely if person A asserts but does not assert truly. Person A asserting falsely is not the same as it being false that person A asserts.

A asserts falsely that Φ *(a, b) =df.*
$Says(A, \Phi, a, b) \bullet (\exists w, u, v)\, R^A(\Phi, w)) \bullet R^A(a, u)) \bullet R^A(b,v)$
$\bullet \sim ((\iota z)(S^A(w, z)))\, \{(\iota x)\, S^A(u, x), (\iota y)\, S^A(v, y)\}$

This requires A to make the statement and have ideas of his words. Otherwise there is not even a false assertion. The assertion is false if there are not objects corresponding to any of the ideas or the corresponding relation does not hold among them.

The Liar.

I think the above account best represents the approach Russell meant to be exploring in his 1906 work notes called "On Substitution". Russell goes on in the work notes to investigate what happens with "A lies". I think he is considering the Liar, where we have a case of a false assertion. He is not considering lying which, of course, involves quite complicated intentions to deceive. I also think he hoped to gain insight to Russell's paradox with this investigation.

If we consider lying in the case of n-ary predicates, we have:

$Lies(A, \Phi, a_1, a_2, \ldots, a_n)$ *=df.*

$(Says(A, \Phi\ , a_1,a_2, \ldots, a_n) \bullet (\exists w, u_1, u_2,\ldots,u_n)\ (R^A\ (\Phi, w)$
$\bullet R^A\ (a_1, u_1) \bullet R^A\ (a_2, u_2) \bullet \ldots. \bullet R^A\ (a_n, u_n))$
$\supset {\sim}((\iota\, z)(\ S^A\ (w, z)))\ \{(\iota\, x_1)\ S^A(u_1,x_1)\ , (\iota\, x_2)\ S^A(u_2,x_2)\ , \ldots, \ \ (\iota\, x_n)$
$S^A(u_n,x_n)\}$

And
Lies(A) =df.
$(\exists\ \Phi\ \ , a_1,a_2, \ldots, a_n)\ Lies(A, \Phi\ \ , a_1,a_2, \ldots, a_n)$

Asserts_truly(A,"lies","A") =
(Says(A,"lies", "A") $\bullet (\exists w, u_1)\ (R^A\ (\text{"lies"}, w)) \bullet R^A\ (\text{"A"}, u_1))$
$\supset {\sim}((\iota z)(\ S^A\ (w, z)))\ \{(\iota x_1)\ S^A(u_1,x_1)\}$

Try *w = i_lies,* u_1 *= i_A,*
Then we get
Says(A, "Lies", "A") $\bullet (R^A\ (\text{"lies", i\_lies})) \bullet R^A(\text{"A", i\_A}))$
$\supset {\sim}((\iota z)(\ S^A\ (\text{i\_lies}, z)))\ \{(\iota x_1)\ S^A(\text{i\_A},x_1)\}$
Using the definition, we get.
Says(A, "Lies", "A") $\bullet (R^A\ (\text{"lies", i\_lies})) \bullet R^A\ (\text{"A", i\_A}))$
$\supset$ *~lies(A)*

So *Asserts_truly(A,"lies","A")* $\supset$ *~lies(A)*.

Similarly *Asserts_falsely(A,"lies","A")* $\supset$ *lies(A)*.

But the Lies in the reasoning is a 3-ary relation. The conclusion is monadic. We produced only, *"lies"* and *"A"* for $\Phi\ \ , a_1,a_2, \ldots, a_n$ that A asserts but is false. We can claim that the idea *i_lies* which would occur in the
Lies(A) =df.
$(\exists\ \Phi\ \ , a_1,a_2, \ldots, a_n)\ Lies(A, \Phi\ \ , a_1,a_2, \ldots, a_n)$
Is a definition distinct from *i_lies* which would occur in

$Lies(A, \Phi\ , a_1, a_2, \ldots, a_n)$ =df.
$(Says(A, \Phi\ , a_1, a_2, \ldots, a_n) \bullet (\exists w, u_1, u_2, \ldots, u_n)\ (R^A\ (\Phi, w)$
$\bullet R^A\ (a_1, u_1), R^A\ (a_2, u_2), \ldots., R^A\ (a_n, u_n))$
$\supset \sim((\iota z)(\ S^A\ (w, z)))\ \{(\iota x_1)\ S^A(u_1, x_1)\ , (\iota x_2)\ S^A(u_2, x_2)\ , \ldots,\ \ (\iota x)\ S^A(u_n, x_n)\}$

And that there is really no contradiction. We only get a contradiction by confusing the definitions of i_Lies with different arities. The two instances of *i_lies* are different! The ideas are distinct. We need to distinguish the ideas in the two definitions of "Lies."

Conclusion on the Liar

The Liar Paradox is resolved, it seems. Russell seemed to have given up on this approach, but he was close. He considered that the Liar was asserting a proposition that, however, did not exist. He said "Then there is a proposition p of which it is asserted that A asserts p and that p is false. But this is not the case. A asserts there is a proposition, but it is not the case that there is such a proposition." I think it a distinct proposition. (in OS, p. 186). He goes on, "*Assume what A asserts to be false.* Then there is no proposition p of which it is asserted that A asserts p and that p is false. This is in fact the case. A does not assert 'I assert p and p is false'; he asserts '*There is* a proposition p such that I assert p and p is false.'" (OS, p. 186) This reminds me of Moore's paradox, I do not know their priority. I think Russell's was on the right track, but he did not analyze the situation fully enough. His mistake was partly in representing the proposition by a single letter – *p*. He recognized that perhaps this was the problem (OS, p. 187). He also recognized a need for symbolism for names of apparent variables, and he does talk about ideas for them (OS p. 188). He talks of the idea of 'everything' having no object – being a *mere* idea – and that the idea of 'something' being similar. It is unfortunate that he did not pursue this approach more fully. My own view is that we must take as real at least one of the quantifiers and one truth function, as well as variables. That is, not only the ideas, but also the corresponding objects. Otherwise, there are no facts to potentially correspond to relations of the ideas to make the beliefs true or false.

Russell's Paradox

Whether these ideas can be extended to solve Russell's paradox is very doubtful.

I will consider the simpler but similar case of Grelling's Paradox (Quine, The Ways of Paradox, pp. 4-5)

Heterological(F) = ~F(F)

A asserts_truly(A, "Heterological", F) =df.

Says(A, "Heterological",F)

• (∃ w) R^A ("Heterological", w) • (∃ x) R^A(F,x)

• ((ιz)(S^A (w, z))) {(ιy) S^A(x, y)}

A asserts_falsely(A, "Heterological", F) =df.

Says(A, "Heterological",F)

• (∃ w) R^A ("Heterological", w) • (∃ x) R^A(F,x)

• ~ ((ιz)(S^A (w, z))) {(ιy) S^A(x, y)}

What if we substitute i_heterological for w and "Heterological" for F, and i_heterological for x we get
heterological(heterological) for *((ιz)(S^A (w, z))) {(ιy) S^A(x, y)} or*
~heterological(heterological) for *~((ιz)(S^A (w, z))) {(ιy) S^A(x, y)}*

Each, by definition, implies the opposite. I think the two instances of "Heterological" must be taken as different, as R is many-one. The obvious way being to adopt a theory of types.

Opacity of Belief

Here I discuss a case of Quine's given in Word and Object, pp. 142-146.

Remember
A believes that $\varphi(a, b)$ =df.
$(\exists x, y, z)\ B_{private}(A, \varphi, a, b) \bullet S^A (a, x)) \bullet S^A (b, y))\ )\ \bullet S^A (\varphi, z)$

Suppose
A = Tom
Φ = Tom's idea of denouncing
a = Tom's idea of Cicero
b = Tom's idea of Catiline
c = Tom's idea of Tully

Remember
A believes that $\varphi(a, b)$ =df.
$(\exists x, y, z)\ B_{private}(A, \varphi, a, b) \bullet S^A (a, x)) \bullet S^A (b, y))\ )\ \bullet S^A (\varphi, z)$

i.e. Tom believes Cicero denounced Catiline.

and
A believes that ~ $\varphi(a, c)$ =df.
$(\exists w, y, z)\ B_{private}\ ({\sim}, A, \varphi, c, b) \bullet S^A (c, w)) \bullet S^A (b, y))\ )\ \bullet S^A (\varphi, z)$

i.e. Tom believes Tully did not denounce Catiline.

But it is true (Unknown to Tom) that Cicero = Tully.

We have $(\exists\, x)\ S^A(a, x)\ \&\ (\exists\, w)\ S^A(c, w)$

It happens that $(\iota\, x)\ S^A(a, x) = (\iota\, w)\ S^A(c, w)$

But it is not logical truth – Tom need not have any reason to believe it.

Definite descriptions, unlike names, do not obey substitutivity of identity.
According to PM (following *14.03) $(\iota\, x)\ S^A(a, x) = (\iota\, w)\ S^A(c, w)$
Is
$(\exists\, b)(x)(\ S^A(a, x) \equiv x = b)\ \&\ (\exists\, d)\ (w)\ (\ S^A(c, w) \equiv w = d)\ \&\ b = d$
This is not itself in the form of an identity statement, so does not imply substitutivity. I believe other cases of apparent opacity can be handled analogously, although they can be much more complex.

Conclusion

The theory presented here also solves the problems associated with false belief and assertion, especially in cases of "non-existent" entities. Russell believed that "On Denoting" solved the problems with belief and assertion concerning "non-existent" entities. However, "On Denoting" requires a primitive predicate for any such "non-existent" object. The theory here requires only the *R, S* and $B_{private}$ relations in addition to the usual logical notions. The apparent problem of opacity of belief is also resolved in a simple case. We are only able to think in ideas. We posit entities to correspond to our ideas, but only know them by description. The only real things we directly know are our ideas or experiences. It seems we also have some innate logical ideas, although we develop those through experience.