Understanding

the Axiom of Reducibility

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I've never felt I properly understood the "Axiom of Reducibility" of Principia Mathematica. That is until tonight. My puzzlement had been the same as Quine's in his introduction to Russell's "Mathematical logic as based on the theory of types". This as in van Heijenoort's From Frege to Goedel. (pp. 151-152) Quine explains "It says that any function is coextensive with what he calls a predicative function: a function in which the types of apparent variables run no higher than the types of the arguments.... The axiom of reducibility is self-effacing; if it is true, the ramification that it is meant to cope with was pointless to begin with. Russell’s failure to appreciate this point is due to his failure to distinguish between propositional functions as notations and propositional functions as attributes and relations.

The explanation came from page xxxv of The Collected Papers of Bertrand Russell, volume 5. “’Gradually’, as he wrote to Jourdain in March 1906, ‘I discovered that to assume a separable φ in φx is just the same, essentially, to assume a class defined by φx, and that non-predicative functions must not be analyzable into a φ and an x.’”

The axiom is

*12.1 |- :(∃f) (x) φx .≡. f ! x

I, and I think others, thought the axiom was saying that if there was such a φ then there was such an f. I think it means there can only be such a φ is there is such a f. How one is to determine whether there is such an f is unclear.

I have read more. I think the above is true. But it should be expressed:

*12.1 $\vdash :(\varphi)(\exists f)\ (x)\ \varphi x\ .\equiv.\ f\,!\,x$

And for any such φ it will either be true or be nonsense. f is a predicative function – i.e. not compound. I think they considered f to be an individual – also an independent entity. (See CPBR 5, p. lxxvii.) The φ is dependent – can be complex – and could (if acceptable) result in contradiction. The problem with this is there is no way given to decide what is acceptable.